Boltzmann Equation With Force Term

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Abstract
In this paper we study the local uniqueness of solutions to the Boltzmann equation with an external force term. Assuming the existence of solutions, we prove a local uniqueness result by applying a fixed point theorem of Hardy–Rogers. The collision operator is assumed to be bilinear, symmetric, and non-continuous, which allows extending previous results that require continuity assumptions. The obtained conditions contribute to the uniqueness theory for Boltzmann-type equations with force terms.
Keywords: Boltzmann equation, force term, fixed point, non-continuous operator, Hardy–Rogers theorem.

1 INTRODUCTION

Consider the following problem: let it be

$$f: [0,T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^+$$

that satisfies the equation distributionally:

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f + t\dfrac{\partial F}{\partial t} \cdot \nabla_v f = Q(f,f), \\ f(0,x,v) = f_0(x,v). \end{cases} \tag{1}$$

Where ,$t \in [0,T]$

$$x = (x_1, x_2, \dots, x_n) \in \Omega \subset \mathbb{R}^n$$

and .$v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$
Here

$$F: [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

it is an external vector field that is supposed to be differentiable with respect to time.
$Q(f,g)(x,v)$is an operator of and we assume that it is bilinear, symmetrical and not continuous.$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$
The operator

$$Q(f,g)(v) = \frac{1}{2}\int_{\mathbb{R}^n}\int_{|w|=1} q(v,\theta)[f(v')g(u') + f(u')g(v') - f(v)g(u) - f(u)g(v)]\,du\,dw,$$

The theory of Boltzmann's equation is a special case of (1), since it is symmetrical, bilinear and not continuous. In this sense, we will use a more general operator than the one used in Boltzmann's equation theory.

Functional space
Consider functional space

$$B = \left\{ f \in L^1([0,T] \times \Omega \times \mathbb{R}^n) : \frac{\partial f}{\partial x_i} \in L^1([0,T] \times \Omega \times \mathbb{R}^n), \frac{\partial f}{\partial v_i} \in L^1([0,T] \times \Omega \times \mathbb{R}^n) \right\}$$

with

$$\| f \|_B = \| f \|_{L^1([0,T]\times\Omega\times\mathbb{R}^n)} + \| \frac{\partial f}{\partial x_i} \|_{L^1} + \| \frac{\partial f}{\partial v_i} \|_{L^1},$$

and

$$\| f \|_{L^1([0,T]\times\Omega\times\mathbb{R}^n)} = \int_0^T \int_\Omega \int_{\mathbb{R}^n} | f(t,x,v) | \; dt\, dx\, dv.$$

We will assume that , and$f_0(x,v) \in Bm(\Omega) < \infty$

$$\int_{\mathbb{R}^n} dv < \infty.$$

Particular cases of the differential equation (1)
a) If , then (1) boils down to$F = 0$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), f(0,x,v) = f_0(x,v),$$

which is the classic Boltzmann equation.
b) If and is not time-dependent (stationary field), then (1) reduces to$F \neq 0$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f = Q(f,f), f(0,x,v) = f_0(x,v),$$

which is the classic differential equation with the term force.
If we assume that solutions exist, we are interested in obtaining local uniqueness from them.

2 State of the art
The purpose of this article is to look for uniqueness conditions for the equation

$$\frac{\partial u}{\partial t} + v \cdot \nabla_x u + F \cdot \nabla_v u + t \frac{\partial F}{\partial t} \cdot \nabla_v u = Q(u,u),$$

with the initial condition .$f(0,x,v) = f_0(x,v)$
Some studies of this problem with the term are found in [1], where existence and uniqueness are shown in local conditions. Asano K. (1987) extends the conditions to global considerations. In [2] Bellomo N. et al. (1989), all have conditions close to equilibrium and conservative force fields. In [3], Galeano R. (2007) existence and uniqueness theorems were proved for as a continuous operator. And in [7], Renjun D. et al.$F \cdot \nabla_v u Q$

Here we will use a fixed-point theorem when the operator is non-continuous of the Hardy–Rogers type [5], presented in [6] by Pathak H. (2018). We consider the theorem a contribution to the theory of uniqueness of solutions in differential equations.
Here we prove the following theorem:

Theorem 1
Be

$$F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

differentiable with respect to time, be it

$$0 < t < \sqrt{\frac{1}{8 + 2k\, m(\Omega) \int_{\mathbb{R}^n} dv}},$$

with ,$k > 0$

$$\int_{\mathbb{R}^n} dv < \infty, m(\Omega) < \infty,$$

and a bilinear, symmetrical, non-continuous operator with$Q(\cdot,\cdot)$ $Q \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.
Then there is a **single local solution** to the problem (1).

3 Development
Be

$$f^{\#}(t,x,v) = f(t, x + vt, v + tF).$$

From this we have to:

$$f(t,x,v) = f^{\#}(t, x - vt, v - Ft),$$
$$\frac{\partial f^{\#}}{\partial t} = \frac{\partial f}{\partial t} + v \cdot \nabla_x f + F \cdot \nabla_v f + t\frac{\partial F}{\partial t} \cdot \nabla_v f = Q(f^{\#}, f^{\#}),$$

then

$$\frac{\partial f^{\#}}{\partial t} = Q(f^{\#}, f^{\#}).$$

Now,

$$f^{\#}(0,x,v) = f(0,x,v) = f_0(x,v),$$

for any , .$x \in \Omega v \in \mathbb{R}^n$
How, then$Q \in L^1(\Omega \times \mathbb{R}^n)$

$$f^{\#}(t,x,v) = f^{\#}(0,x,v) + \int_0^t Q(f^{\#}, f^{\#})\, d\tau. \qquad (3)$$

Let's consider two solutions of the above equation $f_1^{\#}(t,x,v)$ and define$f_2^{\#}(t,x,v)$

$$g^{\#}(t,x,v) = f_1^{\#}(t,x,v) - f_2^{\#}(t,x,v).$$

Then

$$f_1^{\#}(t,x,v) = f_2^{\#}(t,x,v) + g^{\#}(t,x,v),$$

and

$$g^{\#}(0,x,v) = f_1^{\#}(0,x,v) - f_2^{\#}(0,x,v) = 0,$$

I mean

$$g^{\#}(0,x,v) = 0.$$

Since and are solutions of equation (3), it has to be$f_1^{\#}(t,x,v) f_2^{\#}(t,x,v)$

$$f_1^{\#}(t,x,v) = f_1^{\#}(0,x,v) + \int_0^t Q(f_1^{\#}, f_1^{\#})\, d\tau,$$

and

$$f_2^{\#}(t,x,v) = f_2^{\#}(0,x,v) + \int_0^t Q(f_2^{\#}, f_2^{\#})\, d\tau.$$

Then

$$g^{\#}(t,x,v) = \int_0^t Q(f_2^{\#} + g^{\#}, f_2^{\#} + g^{\#})\, d\tau - \int_0^t Q(f_2^{\#}, f_2^{\#})\, d\tau.$$

Let's define the operator in$J$ $L^1([0,T] \times \Omega \times \mathbb{R}^n)$

$$J(g^{\#}) = \int_0^t Q(f_2^{\#} + g^{\#}, f_2^{\#} + g^{\#})\, d\tau - \int_0^t Q(f_2^{\#}, f_2^{\#})\, d\tau.$$

$$J: L^1([0,T] \times \Omega \times \mathbb{R}^n) \longrightarrow L^1([0,T] \times \Omega \times \mathbb{R}^n).$$

Suppose that ,he is the .$J(g^{\#}) \in B \subset L^1([0,T] \times \Omega \times \mathbb{R}^n) 0 \in B 0 \in L^1([0,T] \times \Omega \times \mathbb{R}^n)$
$0$is a fixed point of , since$J$

$$J(0) = \int_0^t Q(f_2^{\#}, f_2^{\#})\, d\tau - \int_0^t Q(f_2^{\#}, f_2^{\#})\, d\tau = 0.$$

Let's verify that zero is the only fixed point of by means of a Hardy–Rogers fixed-point theorem, concluding that$J$
$g^{\#}(t,x,v) \equiv 0$ and therefore
$f_1^{\#}(t,x,v) = f_2^{\#}(t,x,v)$.
Suppose that ,$J(g^{\#}) \in B \subset L^1([0,T] \times \Omega \times \mathbb{R}^n)$
$0 \in B$ is the .$0 \in L^1([0,T] \times \Omega \times \mathbb{R}^n)$
Consider ,$h^{\#}(t,x,v) \geq g^{\#}(t,x,v)$
$h^{\#} \geq g^{\#}$ and evaluate:

$$\begin{aligned} J(g^{\#}) - J(h^{\#}) \quad &= \int_0^t Q(\,f_2^{\#} + g^{\#}, f_2^{\#} + g^{\#})\,d\tau - \int_0^t Q(\,f_2^{\#}, f_2^{\#})\,d\tau \\ &- \int_0^t Q(\,f_2^{\#} + h^{\#}, f_2^{\#} + h^{\#})\,d\tau + \int_0^t Q(\,f_2^{\#}, f_2^{\#})\,d\tau. \end{aligned}$$

This is

$$J(g^{\#}) - J(h^{\#}) \quad = \int_0^t Q(\,f_2^{\#} + g^{\#}, f_2^{\#} + g^{\#})\,d\tau - \int_0^t Q(\,f_2^{\#} + h^{\#}, f_2^{\#} + h^{\#})\,d\tau.$$

Developing,

$$\begin{aligned} J(g^{\#}) - J(h^{\#}) \quad &= \int_0^t Q(\,f_2^{\#}, f_2^{\#})\,d\tau + \int_0^t Q(\,f_2^{\#}, g^{\#})\,d\tau + \int_0^t Q(\,g^{\#}, f_2^{\#})\,d\tau + \int_0^t Q(\,g^{\#}, g^{\#})\,d\tau \\ &- \int_0^t Q(\,f_2^{\#}, f_2^{\#})\,d\tau - \int_0^t Q(\,f_2^{\#}, h^{\#})\,d\tau - \int_0^t Q(\,h^{\#}, f_2^{\#})\,d\tau - \int_0^t Q(\,h^{\#}, h^{\#})\,d\tau. \end{aligned}$$

So,

$$\begin{aligned} J(g^{\#}) - J(h^{\#}) \quad &= 2\int_0^t Q(\,f_2^{\#}, g^{\#})\,d\tau + \int_0^t Q(\,g^{\#}, g^{\#})\,d\tau \\ &-2\int_0^t Q(\,f_2^{\#}, h^{\#})\,d\tau - \int_0^t Q(\,h^{\#}, h^{\#})\,d\tau. \end{aligned}$$

The above is true since

$$Q(2f_2^{\#}, g^{\#}) = Q(f_2^{\#} + f_2^{\#}, g^{\#}) = Q(f_2^{\#}, g^{\#}) + Q(f_2^{\#}, g^{\#}) = 2Q(f_2^{\#}, g^{\#}).$$

Then,

$$J(g^{\#}) - J(h^{\#}) \quad = \int_0^t Q(2\,f_2^{\#}, g^{\#} - h^{\#})\,d\tau + \int_0^t Q(\,g^{\#} - h^{\#}, g^{\#} + h^{\#})\,d\tau.$$

This is

$$J(g^{\#}) - J(h^{\#}) = -\int_0^t Q(2\,f_2^{\#}, g^{\#} - h^{\#})\,d\tau - \int_0^t Q(\,g^{\#} - h^{\#}, g^{\#} + h^{\#})\,d\tau.$$

Since it is a non-continuous operator, then there is a such that$Qk > 0$

$$Q(h^{\#} - g^{\#}, p) > k(h^{\#} - g^{\#}),$$

for any $.p \in B$

For any one you have to$p \in B$

$$-\,Q(h^{\#} - g^{\#}, p) < k(g^{\#} - h^{\#}),$$

then

$$J(g^{\#}) - J(h^{\#}) < k\int_0^t (g^{\#} - h^{\#})\,d\tau + k\int_0^t (g^{\#} - h^{\#})\,d\tau.$$

Therefore,

$$|J(g^{\#}) - J(h^{\#})| < k\int_0^t |g^{\#} - h^{\#}|\;d\tau + k\int_0^t |g^{\#} - h^{\#}|\;d\tau < 2Tk\int_0^t |g^{\#} - h^{\#}|\;d\tau.$$

So,

$$\| J(g^{\#}) - J(h^{\#}) \|_{L^1([0,T]\times\Omega\times\mathbb{R}^n)} < 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv[\| h^{\#} - J(h^{\#}) \|_{L^1} + \| J(h^{\#}) - J(g^{\#}) \|_{L^1} + \| J(g^{\#}) - g^{\#} \|_{L^1}].$$

I mean

$$\begin{aligned}\| J(g^{\#}) - J(h^{\#}) \|_{L^1} &< 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv\;\| h^{\#} - J(h^{\#}) \|_{L^1}\\ &+2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv\;\| J(h^{\#}) - J(g^{\#}) \|_{L^1}\\ &+2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv\;\| J(g^{\#}) - g^{\#} \|_{L^1}.\end{aligned}$$

Then,

$$\left(1 - 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv\right) \| J(g^{\#}) - J(h^{\#}) \|_{L^1} < 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv(\| h^{\#} - J(h^{\#}) \|_{L^1} + \| J(g^{\#}) - g^{\#} \|_{L^1}).$$

Therefore,

$$\| J(g^{\#}) - J(h^{\#}) \|_{L^1} < \frac{2kT^2}{1 - 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv} \| h^{\#} - J(h^{\#}) \|_{L^1} + \frac{2kT^2}{1 - 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv} \| J(g^{\#}) - g^{\#} \|_{L^1}.$$

According to the Hardy–Rogers fixed-point theorem, making

$$a = \frac{4T^2}{1 - 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv}, b = 0, c = 0,$$

then

$$2a = \frac{8T^2}{1 - 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv} < 1$$

If

$$8T^2 < 1 - 2kT^2 m(\Omega)\int_{\mathbb{R}^n} dv,$$

this is

$$8T^2 + 2kT^2 m(\Omega) \int_{\mathbb{R}^n} dv < 1.$$

Then,

$$T^2 < \frac{1}{8 + 2k\, m(\Omega) \int_{\mathbb{R}^n} dv}, T < \sqrt{\frac{1}{8 + 2k\, m(\Omega) \int_{\mathbb{R}^n} dv}}.$$

We have the existence of a **single fixed point** of , and since it is a fixed point, it is concluded that$J0$

$$g^{\#}(t, x, v) \equiv 0, \forall t \in [0, T],\ x \in \Omega,\ v \in \mathbb{R}^n.$$

Therefore,

$$f_1^{\#}(t, x, v) = f_2^{\#}(t, x, v), t \in [0, T],\ x \in \Omega,\ v \in \mathbb{R}^n.$$

CONCLUSIONS

In this work, we established a local uniqueness result for solutions of the Boltzmann equation with an external force term. The analysis is based on a fixed point approach of Hardy–Rogers type, which allows treating collision operators that are bilinear, symmetric, and non-continuous.

The obtained result extends previous uniqueness theorems that rely on continuity assumptions for the collision operator. This framework may be useful for further studies on Boltzmann-type equations with more general force fields or non-standard collision kernels. Future work could address global uniqueness or existence results under weaker assumptions.

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